Extended Essay in group 5

Mathematics

Title :

The existing relationship between π and prime number distribution

Research question :

What is the relationship between π, the Leibniz series and the distribution of prime numbers?

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Contents

1 INTRODUCTION

The π constant is a number that mathematicians and scientists find everywhere in many fields, such as probability, astronomy and quantum physics. One of its appearances that seems strange is its appearance in number theory with the distribution of prime numbers, which leads to the question "What is the relationship between π, the Leibniz series and the distribution of prime numbers?". This essay will therefore explore the role of the distribution of prime numbers to express π and also the relationship between these two. This essay will also explore some of the properties of π such as its irrationality, Archimedes' approximation of π and also infinitesimal calculus, notably the Leibniz series, to establish preliminary knowledge to explore Grant Sanderson's method, which was able to link π and the distribution of prime numbers. This research question is relevant because it allows us to see π through a new point of view. In addition, the distribution of prime numbers is a contemporary subject that is the essence of cryptography that is under development. In other words, this investigation contributes to the exploration of the distribution of prime numbers.

I chose this subject because I would like to have a career as a computer engineer, especially in cybersecurity and cryptography, so this exploration will play the role of an introduction to the field of cryptography but from a different point of view than the one introduced in universities.

2 PRIMARY KNOWLEDGE

2.1 Primary Knowledge on π

The constant π, also known as the "Archimedes' Constant" is the ratio between the circumference of a circle and its diameter.

$$\pi = \frac{C}{d} \tag{2.1}$$

Several ancient civilizations have approximated this constant, such as the Egyptians who approximated it at $\pi = \frac{2^8}{3^4} = 3.1605\ldots$ and the Babylonians at $\pi = 3+\frac{1}{8} = 3.125$. Despite its presence in nature, this number is not so rudimentary; it is in fact an irrational number, a number that cannot be expressed as a quotient between two integers. This number can also be used to determine the power of a computer. Trillions of decimals of π have been calculated over the last century.[18] What interests me in this number is that because of these appearances in many areas, I think there will also be other areas in which it will appear; it may be related to contemporary problems.

2.2 The beginning of infinitesimal calculus

Infinitesimal calculus dates back to antiquity, but the daily notation was put by Goddfrey Leibniz during the late 17th century. The latter is considered, along with Newton, to be the father of infinitesimal calculus. The idea is relatively simple; to establish a link between the variation of several variables and to measure the area and volume of objects by studying the effect of an "infinitely small" change. In other words, how a very small change of x affects $f(x)$, or, calculate the air under a curve using rectangles whose width is very small; the smaller the width, the more accurate the result. [8] However, this concept has existed since antiquity as Archimedes applied this idea to be able to approximate π.

2.3 Archimedes' Approximation

Archimedes managed to frame π between two rational numbers using a geometric proof:

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7} \tag{2.2}$$

He started by creating two hexagons, one circumscribed and the other inscribed in a circle with a diameter of r. Since π is the ratio between the circumference and the diameter, then the ratio between the perimeter and the diameter of the inscribed polygon will necessarily be smaller than π but necessarily larger for the circumscribed polygon, which will allow him to frame π between two numbers.

By increasing the number of sides of these polygons, the difference between the ratio will be smaller and then the interval in which we are sure that π exists will be smaller. In other words, when the number of sides of this polygon approaches infinity, the ratio and interval will approach π.

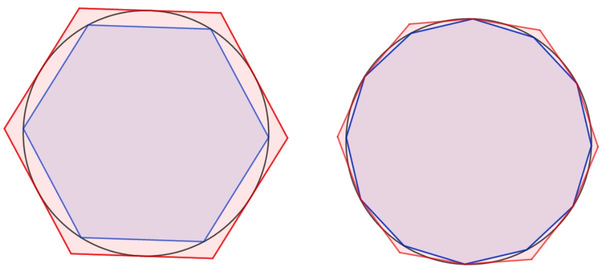

Figure 1: Inscribed and circumscribed polygons. Polygons with 6 sides for the left figure and 12 for the right

However, what Archimedes introduced is that he did not need to measure the perimeter of the polygons each time; he was able to establish a relationship between a n side polygon and a $2n$ side polygon. First, a relationship must be established between the circumscribed polygon of n sides and the $2n$ sides (see Figure 2). Let AB be the radius of a circle with BC the tangent to the point B and, in the case of a hexagon, $C\widehat{A}B = 30°$ with AD as the bisector of this angle.

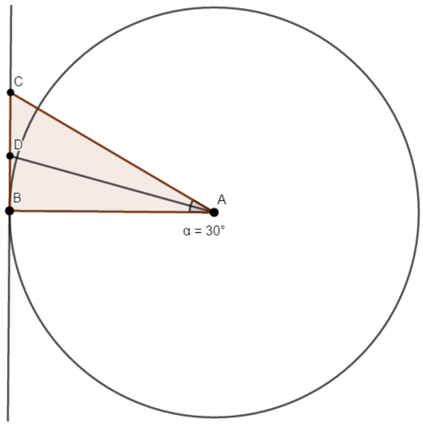

Figure 2: Doubling the sides of a circumscribed polygon

According to proposal 3 of Book VI of *The Elements* by Euclid, we have:

$$\frac{AC}{AB} = \frac{CD}{DB} \implies \frac{CA + AB}{AB} = \frac{CD + DB}{DB} = \frac{CB}{DB} \tag{2.3}$$

multiplying by AB then dividing by CB gives

$$\frac{CA + AB}{CB} = \frac{AB}{DB} \tag{2.4}$$

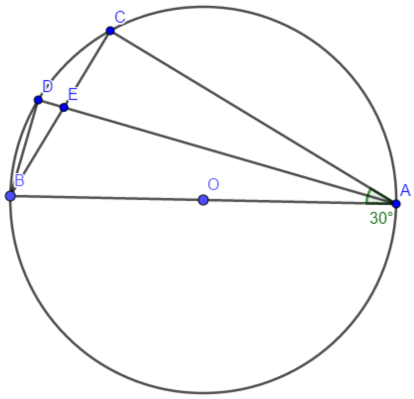

Figure 3: Doubling the sides of an inscribed polygon

Secondly, for the inscribed polygon (see Figure 3) with AD the bisector of the triangle, Archimedes established this relationship:

$$\frac{AB + AC}{BC} = \frac{AD}{BD} \tag{2.5}$$

By repeating the bisector several times (i.e. the $B\widehat{A}D$ bisector in this case), we can then find the polygon perimeter by multiplying the "DB" of each new iteration by n. Finally, Archimedes found a recurrence relationship for a_{2n} and b_{2n}, which are the perimeters of the polygons circumscribed and inscribed with respectively:

$$a_{2n} = \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad b_{2n} = \sqrt{a_{2n} b_n} \tag{2.6}$$

All that remained was to substitute for $a_6 = 4\sqrt{3}$ and $b_6 = 6$ (taking an example of a circle with a radius of 1) and calculate the ratio between the perimeter and the diameter of the successive polygons. Archimedes went as far as a 96-sided polygon to frame π between the values mentioned at the beginning of this section. What is important to remember from this part is that Archimedes created a sequence to approximate π without having to draw many polygons and this sequence becomes more and more precise by increasing iterations [2] [11] [19].

What I see impressive in this method is that even if Archimedes did not have the exact value of π, he was able to find a way to approximate it with such precision that it was sufficient for the needs of his time, as being the ratio between the circumference and the diameter of a circle.

2.4 The irrationality of π

Ironically, the constant which is the ratio between 2 lengths cannot be expressed by a ratio between two integers. The irrationality of π was known since antiquity but no one could prove it before Lambert in 1761. His proof is considered one of the most intuitive and simple proofs that proves the irrationality of π. The essential idea on which he based his proof is the following: "whenever any arc of a circle is commensurable to the radius, the tangent of that arc is immeasurable to him; and conversely, no commensurable tangents are those of a commensurable arc"[13]. In other words, in the first part of the proof, Lambert proved that the tangent of any rational number is irrational. In the beginning, he expressed the tangent function as a continuous fraction, which was quite popular at the time. He began with:

$$\tan x = \frac{\sin x}{\cos x} \tag{2.7}$$

With (using Taylor's expansion at $x = 0$):

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{2.8}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{2.9}$$

Replacing in the tangent equation gives us:

$$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} \tag{2.10}$$

We factor the numerator by x, then dividing this x by the inverse of what remains gives us:

$$\tan x = x \frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} = \frac{x}{\frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} \tag{2.11}$$

Add and subtract the denominator (from the denominator) to the numerator (from the denominator) to eliminate the 1 and thus be able to factor by x^2 (after simplification):

$$\tan x = \frac{x}{\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ - \frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} = \frac{x}{1 - x^2 \frac{\frac{1}{3} - \frac{x^2}{5\times 3!} + \frac{x^4}{7\times 5!} - \frac{x^6}{9\times 7!} + \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}} \tag{2.12}$$

By changing the form once more we get:

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{\cfrac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots}{\frac{1}{3} - \frac{x^2}{5\times3!} + \frac{x^4}{7\times5!} - \frac{x^6}{9\times7!} + \cdots}}} \tag{2.13}$$

To eliminate the constant at the numerator, the denominator is added and subtracted 3 times to obtain:

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3(\frac{1}{3} - \frac{x^2}{5\times3!} + \frac{x^4}{7\times5!} - \frac{x^6}{9\times7!} + \cdots) \cfrac{- \frac{2x^2}{5\times3!} + \frac{4x^4}{7\times5!} - \frac{6x^6}{9\times7!} + \cdots}{\frac{1}{3} - \frac{x^2}{5\times3!} + \frac{x^4}{7\times5!} - \frac{x^6}{9\times7!} + \cdots}}} \tag{2.14}$$

Which simplifies to:

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - x^2 \cfrac{\frac{1}{15} - \frac{x^2}{35\times3!} + \frac{x^4}{63\times5!} - \cdots}{\frac{1}{3} - \frac{x^2}{5\times3!} + \frac{x^4}{7\times5!} - \frac{x^6}{9\times7!} + \cdots}}} \tag{2.15}$$

Repeating these steps will allow Lambert to express the tangent function as a continuous fraction:

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}} \tag{2.16}$$

As with Taylor series, this fraction tends towards the tangent function as more terms are included. In the second part, Lambert managed to prove this equality by finding the general form but this is not in the scope of this essay. In the third part, Lambert proved by contradiction that any rational number $\frac{u}{v}$, $\tan\frac{u}{v}$ is irrational. However, $\tan\frac{\pi}{4} = 1$ (which is rational), which implies that $\frac{\pi}{4}$ cannot be written as a fraction $\frac{a}{b}$ with $a, b \in \mathbb{Z}^+$. Which implies that π is irrational.[16] [13] I see that this demonstration also shows the flexibility of mathematics because it uses the continued fractions to prove the irrationality of a number that is originally defined in geometry, which shows that the different mathematical branches are complementary.

To express an irrational number, mathematicians have several choices. An irrational number can be expressed as an infinite product as the one that converges to π discovered in 1655 by John Wallis [20],

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\times\frac{2n}{2n+1}\right) = \frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\cdots = \frac{\pi}{2} \tag{2.17}$$

or as a continuous fraction as discovered by William Brouncker in the 1660s, [12]

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}} \tag{2.18}$$

or in the form of series such as Leibniz's which will be explored in the next section.

2.5 Leibniz's Series

This series is also called Madhava-Leibniz because it was discovered during the 14th century by Madhava and also independently discovered by Leibniz in the 1670s.

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{2.19}$$

This series will be used in this essay and so I see the need to explore this series. What I see as relevant and interesting in this series is the geometric demonstration that Leibniz gave. His demonstration consists of finding another expression for the area under the quarter of a circle of radius 1, (which is of measure $\frac{\pi}{4}$). Leibniz, therefore, as the father of integration, chose to integrate a quarter circle with a radius of $OA = 1$. Leibniz noticed that he only needs integrate the area between the OT segment and the OT bow because the area of the $OAT = 1/2$ triangle.

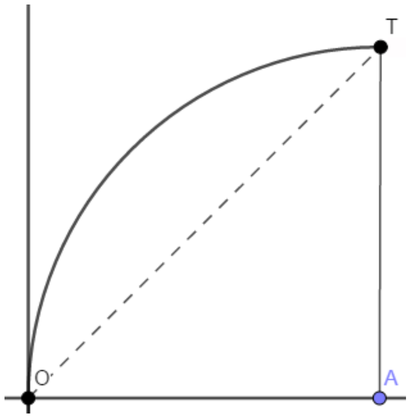

Figure 4: Quarter of a circle divided into a triangle and an area C (bounded by the bow OT and the rope OT)

Let's consider the OPQ triangle (see figure 5). Leibniz's goal is to integrate OPQ throughout C. Let P and Q be two points infinitely close on the arc with a distance $PQ = ds$. Then, let OR be the height from P, S the intersection between Oy and QR and OPQ a triangle with an infinitesimal area.

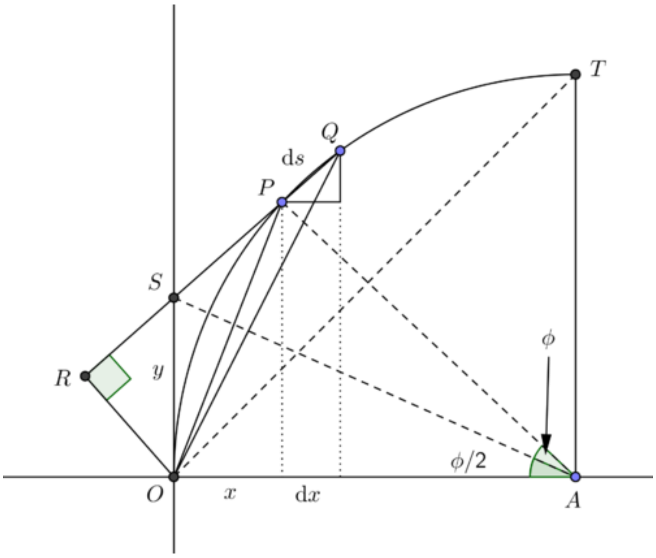

Figure 5: Quart d'un cercle sur un repère [4]

We can therefore express dC as the area of OPQ

$$dC = \frac{OR \times PQ}{2} = \frac{OR \times ds}{2} \tag{2.20}$$

Leibniz also noticed that ORS and the small triangle on PQ are similar ($\widehat{P} = \widehat{O}$, $\widehat{Q} = \widehat{S}$ and they have a right angle), therefore

$$\frac{ds}{dx} = \frac{OS}{OR} \iff OR \times ds = OS \times dx \tag{2.21}$$

Therefore

$$dC = \frac{OS \times dx}{2} = \frac{ydx}{2} \tag{2.22}$$

For $y = f(x) = OS$. Then Leibniz noted the abscissa of $P = x$ and integrated the following

$$C = \frac{1}{2}\int_0^1 ydx \tag{2.23}$$

After the integration by parts we get

$$C = \left[\frac{1}{2}xy\right]_0^1 - \int_0^1 \frac{x}{2}dy = \frac{1}{2} - \int_0^1 \frac{x}{2}dy \tag{2.24}$$

It is necessary to note graphically for $x = 1$, $y = 1$. SPA and SOA are isometric, because $OA = AP$ (radius of the circle), $\widehat{O} = \widehat{P} = 90°$ and therefore

$$OS^2 = SA^2 - OA^2 = SA^2 - AP^2 = SP^2 \implies OS = SP \tag{2.25}$$

Which implies that SA is the bisector of the agnle ϕ. Leibniz looked for a way to link x and y and used trigonometry for this. Leibniz noticed that $y = \tan\frac{\phi}{2}$ and $x = 1 - \cos\phi =$

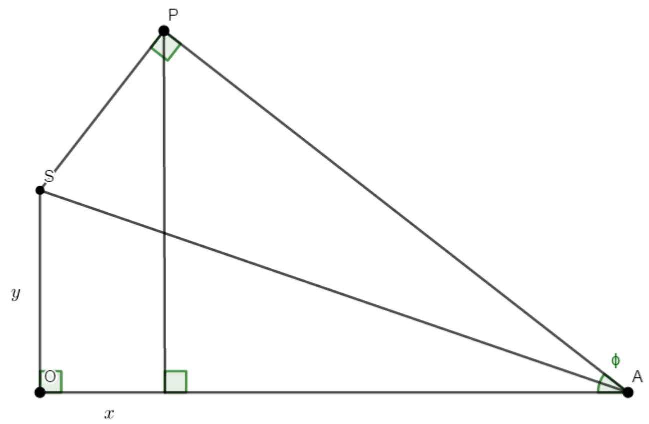

Figure 6: Trigonometric relationship between x and y

$2\sin^2\frac{\phi}{2}$ (see Figure 6), by manipulating the formulas we get:

$$\tan^2\frac{\phi}{2} = \frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}} = \sin^2\frac{\phi}{2}\sec^2\frac{\phi}{2} = \sin^2\frac{\phi}{2}\left(1 + \tan^2\frac{\phi}{2}\right) \tag{2.26}$$

By replacing with x and y we get:

$$y^2 = \frac{x}{2}\left(1 + y^2\right) \iff \frac{x}{2} = \frac{y^2}{1 + y^2} \tag{2.27}$$

This doesn't seem like much, but it's actually the sum of a geometric series with $U_1 = y^2$ and a common ratio $r = -y^2$. In other words, it means

$$\frac{y^2}{1 + y^2} = y^2 - y^4 + y^6 - y^8 + \cdots \tag{2.28}$$

By replacing the equations 2.27 and 2.28 in the equation 2.24 we get:

$$C = \frac{1}{2} - \int_0^1 \frac{y^2}{1+y^2}dy = \frac{1}{2} - \int_0^1 \left(y^2 - y^4 + y^6 - y^8 + \cdots\right) dy \tag{2.29}$$

$$= \frac{1}{2} - \left[\frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \frac{y^9}{9} + \cdots\right]_0^1 \tag{2.30}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \tag{2.31}$$

By adding the air of the triangle OAT to C, we obtain the Leibniz series which converges at $\frac{\pi}{4}$ (see equation 2.19). What I see relevant in this demonstration is how Leibniz established the relationships between a quarter of a circle, geometry, trigonometry and integration to show that there is a relationship between π and the alternate sum of the inverse of all odd numbers[9] [4] [14].

However, this series is rarely used to approximate the value of π because it converges very slowly, to calculate the value of π correctly to 100 decimal digits we need 10^{100} terms [1]. However, there is one property that is interesting in the Leibniz series; even if there is an error in one digit the function continues with several other correct digits, let us take the example of the sum of the first 50 000 terms

$$4 \sum_{k=0}^{50000} \frac{(-1)^k}{2k+1} = 3.14157\underline{7}26535897\underline{9}52384626\underline{4}238327950\ldots$$

$$\text{with } \pi = 3.14159265358979323846264338327950\ldots$$

The underlined numbers are the incorrect numbers. This property was discovered in 1987 using computers. Brothers Borwein and K. Dilcher were able to find the sum of all terms after the $N/2^{\text{ème}}$ to be able to calculate errors in the series with N divisible by

$$\pi - 4\sum_{k=0}^{N/2} \frac{(-1)^k}{2k+1} \sim \sum_{m=0}^{\infty} \frac{2 \times E_{2m}}{N^{2m+1}} \tag{2.32}$$

$$= \frac{2}{N} - \frac{2}{N^3} + \frac{10}{N^5} - \frac{122}{N^7} + \frac{2770}{N^9} - \dots \tag{2.33}$$

and E_{2m} are the Euler pairs numbers, which are a sequence of integers that also appears in combinatorial mathematics and also in some Taylor series expansions. In the previous example, the value of N is 10^5 and therefore by replacing in the equation 2.33,

$$\frac{2}{10^5} - \frac{2}{10^{15}} + \frac{10}{10^{25}} - \frac{122}{10^{35}} + \frac{2770}{10^{45}} - \dots \tag{2.34}$$

This means that there is an error in the 5^{the} digit after the decimal point of $+2$, the 15^{th} of –2, the 25^{th} of $+10$, etc [3] [1]. This development therefore makes it possible to better approximate π in a much more reasonable time.

3 RELATIONSHIP BETWEEN π AND PRIME NUMBER DISTRIBUTION

Grant Sanderson, known on the Internet as "3Blue1Brown", released a video on May 19, 2017 entitled "Pi hiding in prime regularities [17]" in which he shows the relationship between π, complex numbers and prime numbers. The purpose of his video is to show how several tools from different origins (π with an origin in geometry, primes in number theory and complexe numbers in algebra) are related in modern mathematics, especially in number analytic theory, a branch that uses analysis to solve discrete math problems. The method he used consisted in finding a systematic way to calculate the area of a circle on the complex plane.

3.1 Measuring the area of a circle

Sanderson is convinced that when π appears in a series or equation, there is always a hidden circle. So he started by finding a different way to measure the area of a circle with a radius of R. A primitive way is to count the tiles in the circle but this is very imprecise when the radius is small. However, at the limit, when the radius approaches infinity, the number of squares will approach more and more towards the area of the circle. In other words, Sanderson calculated the area of the circle using the same concept that Archimedes used to approximate π, which is also the same concept of infinitesimal calculation, but applied differently.

Another assumption Sanderson made is to replace counting tiles by counting the number of lattice points, the intersection points between the grids (see Figure 7), which creates uncertainty in circles with a small radius, but with a fairly large radius, the number of lattice points, the number of unit squares and the area will approach each other.

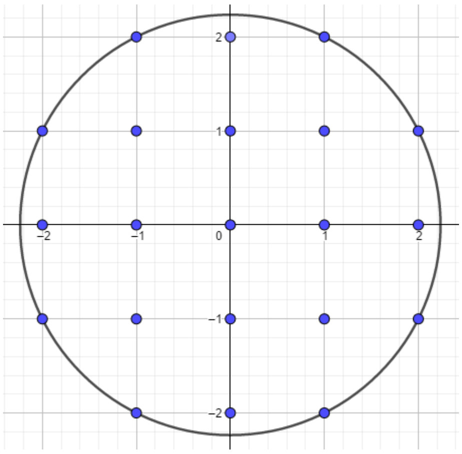

Figure 7: Circle with a radius of $\sqrt{5}$ and lattice points

In other words, the lattice points are the coordinate points $(x; y)$ with $x, y \in \mathbb{Z}$. This shows that the distance between a lattice point and the origin is $\sqrt{x^2 + y^2}$. Sanderson was more specifically interested in the lattice points that are on the circle, because for a D disc with a radius of $\sqrt{N}$ with $N \in \mathbb{Z}^+$

$$\text{Lattice points in } D = \sum_{i=0}^{N} \text{Lattice points on a circle with raidus } \sqrt{i} \tag{3.1}$$

Here is the table of the first lattice points of the radius circles $\sqrt{N}$ for $N \in \mathbb{Z}^+$ and P the number of lattice points on the circle.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
P	4	4	0	4	8	0	0	4	4	8	0	0	8	0	0	4	8

Table 1: List of the number of lattice points according to the radius of the circle

The numbers of lattice points seem a little arbitrary (see Table 1). Yet, according to Sanderson, it is in these cases that mathematics proves its beauty. For example, a circle with a radius of $\sqrt{5}$ (see Figure 7) will pass through the coordinates (2;1), (1;2), (−1;2), (−2;1), etc.

3.2 On the complex plane

According to Sanderson, when there is a problem on the Euclidean plane, it is always fruitful to think about it on the complex plane. This will make it possible to manipulate the coordinates even better to find a systematic way to count the number of lattice points. Let's take again the example of the radius circle $\sqrt{5}$ with $5 = 2^2 + 1^2$. This can therefore be factorized to $(2+i)(2-i)$, $(1+2i)(1-2i)$, etc. In this case, the lattice points can be expressed by being "Gaussian Integers", which are defined as the set

$$\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\} \tag{3.2}$$

Working with Gaussian integers is similar to working with complex numbers, the only difference in the definition is the norm N which is defined [5]

$$N(a + bi) = (a + bi)(a - bi) = a^2 + b^2 \tag{3.3}$$

3.3 Prime number factorization

Some primes in $\mathbb{N}$ such as 5,13,17 and 29 are not primes in the Gaussian integers because they can be factorized using complex numbers. These prime numbers are numbers that can be expressed by being the sum of the two squares. In other words, for some prime numbers p

$$p = a^2 + b^2 = (a + bi)(a - bi) = (b + ai)(b - ai) \tag{3.4}$$

However, other prime numbers remain unfactorizable, such as 3, 11, 19 and 23, even in the complex plane. This property of prime numbers that can be factorized using complex integers will play a large role in exploring and finding the relationship between π and these prime numbers. However, we can notice that circles with a radius of $\sqrt{3}$ or

$\sqrt{11}$, or other prime non-factorizable numbers, do not pass over any lattice points. Several mathematicians have explored this property like Fermat, who established a theorem to recognize these numbers.

3.4 Fermat's theorem on sum of two squares

This theorem of Fermat's two squares states that for a prime number $p > 2$ and $a, b, k \in \mathbb{Z}^+$

$$p = a^2 + b^2 \iff p = 4k + 1 \tag{3.5}$$

Fermat did not prove this theorem and the first demonstration was that of Euler but I found it complicated and requires a high level of mathematical maturity and so I looked for a less complicated and more appropriate demonstration for this essay: the second demonstration by Richard Dedekind, a great mathematician of the 19th century, who uses Gaussian integers.

The direction $\Rightarrow$ is easy to prove: a perfect square x can be of the form $4q^2$ if x is even or $4(q^2 + q) + 1$ if it is odd and therefore the sum of the two perfect squares modulo 4 can only be 0, 1 or 2. In other words, for an odd number which is the sum of the two squares, it must be of the form $4k + 1$.

The other direction $\Leftarrow$ is more difficult to prove and that's where Dedekind demonstrated this assertion by using Gaussian integers. He began by showing that a prime number $p = 4k + 1$ can divide $m^2 + 1$. This is equivalent to saying: Is there a number m^2 that gives a remainder of $p - 1$ when divided by p? In other words, in modular arithmetic notation

$$m^2 \equiv -1 \pmod{p} \tag{3.6}$$

Fortunately, the Euler Criterion [6] is a criterion that allows to know for an integer q

relatively prime to p

$$q^{\frac{p-1}{2}} \equiv \begin{cases} 1 \pmod p \text{ if there is an integer } m \text{ where } q \equiv m^2 \pmod p \\ -1 \pmod p \text{ Ohterwise} \end{cases} \tag{3.7}$$

Substituting for $q = -1$ and $p = 4k + 1$ gives

$$(-1)^{\frac{4k+1-1}{2}} = (-1)^{2k} = 1 \tag{3.8}$$

Which means that $p|m^2 + 1$ with $m^2 + 1 = (m + i)(m - i)$. However, p does not divide the factors of $m^2 + 1$ (it does not divide the imaginary parts) so p is not prime in the Gaussian integers and can therefore only be factorized. The norm of a Gaussian integer is multiplicative ($N(n)N(m) = N(mn)$, as in $\mathbb{C}$) and therefore

$$N(p) = N(x_1)N(x_2)\dots N(x_i) = p^2 \tag{3.9}$$

$N(x_i)$ can only be equal to p or p^2 (a single factorization does not include $N(x_i) = 1$). But there are at least 2 factors and therefore having a norm of p^2 is a contradiction and then there are exactly 2 factors in the form of $p = (a + bi)(a - bi) \implies p = a^2 + b^2$[7]. On the other hand, if $p = 4k + 3$, it is possible to prove that there are no integers m for $m^2 \equiv -1 \pmod p$ using the Euler criterion.

$$(-1)^{\frac{4k+3-1}{2}} = (-1)^{2k+1} = -1 \tag{3.10}$$

This allowed me to understand and served to reinforce my knowledge of an argument with the form “if and only if”.

3.5 Estimation of the number of lattice points

A prime number in the form of $4k+1$ can be factorized in 2 different ways $(a+bi)(a-bi)$ and $(b+ai)(b-ai)$. However, these lattice points also exist according to the axial symmetries of O_x and O_y. In other words, if $a+bi$ is on the circle, then $ai-b$, $-a-bi$ and $-ai+b$ are also on the circle; we can just multiply by 1, i, –1 and $-i$ respectively.

In his presentation, Sanderson explored the properties of these numbers to be able to calculate the numbers of lattice points only with the decomposition of the number into prime factors.

1. The prime numbers in the form of $4k+3$ must have even power to have lattice points.
2. Multiplying by powers of 2 does not change the number of lattice points.
3. Each prime number of form $4k+1$ will add a lattice point that will have its cousins multiplied, which can be obtained my multiplying by i, $-i$ and –1.
4. Each first number p^n of form $4k+1$ adds $4(n+1)$ lattice points.
5. To find the number of lattice points, simply multiply the number of lattice points added for each prime number.

Properties 1 to 3 have been rigorously proven on $p = a^2 + b^2$ by Euler [10] and 4 to 5 were noticed by Sanderson. It is worth noting that the reason why multiplying by 2 does not change the number of lattice points is interesting. $2 = (1+i)(1-i)$ and therefore the angle between the factors of 2 is a right angle. In other words, we can simply multiply by i or $-i$ to get the other factor, which is already taken into account above.

Let's take the example of $50 = 2 \times 5^2$. Depending on these properties, there are 4(3) =12 lattice points. Another example $49725 = 5^2 \times 3^2 \times 3^2 \times 13 \times 17$ will therefore have $4(3)(2)(2)(2) = 48$ lattice points. However $1100 = 2^2 \times 5^2 \times 11$ has no lattice points, because it contains a prime number of form $4k + 3$ with an odd power. So radius circles $\sqrt{50}$ or $\sqrt{49725}$ will have lattice points but not $\sqrt{1100}$.

3.6 Dirichlet Character

To be able to systematize the number of lattice points, it is sufficient to use a tool present in analytic number theory: the Dirichlet character modulo 4. This character is nothing more than a multiplicative function defined as

$$\chi(n) = \begin{cases} 0, \text{for } n \text{ even} \\ 1, \text{for } n = 4k + 1 \\ -1, \text{for } n = 4k + 3 \end{cases} \tag{3.11}$$

Taking the example of $450 = 2 \times 3^2 \times 5^2$, we can see that there are $4(1)(1)(1)(3)$ lattice points. What Sanderson noticed is that we can also substitute these prime factors with χ from all powers up to the present power, so to find the number of lattice points in 450, we can add the options like this $4(\chi(1) + \chi(2))(\chi(1) + \chi(3) + \chi(3^2))(\chi(1) + \chi(5) + \chi(5^2)) = 4(1 + 0)(1 - 1 + 1)(1 + 1 + 1)$. Taking another example for $45 = 3^2 \times 5^2$ with the number of lattice points $4(\chi(1) + \chi(3) + \chi(3^2))(\chi(1) + \chi(1) + \chi(5) + \chi(5^2))$, by expanding and because χ is multiplicative, we can simplify to $4(\chi(1) + \chi(3) + \chi(5) + \chi(9) + \chi(15) + \chi(45)$. So for a number $n = p_1^k \times p_2^l \times \cdots \times p_q^m$, the number of lattice points, $4(\chi(p_1^0) + \chi(p_1^1) + \cdots + \chi(p_1^k))(\chi(p_2^0) + \chi(p_2^1) + \cdots + \chi(p_2^l)) \ldots (\chi(p_q^0) + \chi(p_q^1) + \cdots + \chi(p_q^m))$, is the sum of the χ functions of all the divisors of n multiplied by 4.[15]

3.7 Summary

Let's go back to the original problem: find a systematic method to calculate the area of a circle other than πR^2 to give an equivalence relationship when taking the limit as the raidus tends to infinity. Sanderson's method calculates the number of lattice points because it is equivalent to the area when taking the limit. Just add the lattice point numbers for $\sqrt{n}$ with $n \in \mathbb{Z}^+$ up to R^2. This is the same as applying χ to all the divisors of n and then adding everything together and multiplying by 4. When R^2 tends to infinity, we have to count how many times $\chi(n)$ appears. For $n = 1$ it appears R^2 times, $\chi(2)$ appears $R^2/2$ times, $\chi(3)$ appears $R^2/3$ times, etc. It should not be forgotten that this only becomes accurate when $R^2 \to \infty$ (see Figure 8).

$$
\begin{array}{l}
\sqrt{1} \Rightarrow 4\big(\chi(1)\big) \\
\hline
\sqrt{2} \Rightarrow 4\big(\chi(1)+\chi(2)\big) \\
\hline
\sqrt{3} \Rightarrow 4\big(\chi(1)+\chi(3)\big) \\
\hline
\sqrt{4} \Rightarrow 4\big(\chi(1)+\chi(2)+\chi(4)\big) \\
\hline
\sqrt{5} \Rightarrow 4\big(\chi(1)+\chi(5)\big) \\
\hline
\sqrt{6} \Rightarrow 4\big(\chi(1)+\chi(2)+\chi(3)+\chi(6)\big) \\
\hline
\sqrt{7} \Rightarrow 4\big(\chi(1)+\chi(7)\big) \\
\hline
\sqrt{8} \Rightarrow 4\big(\chi(1)+\chi(2)+\chi(4)+\chi(8)\big) \\
\hline
\sqrt{9} \Rightarrow 4\big(\chi(1)+\chi(3)+\chi(9)\big) \\
\hline
\sqrt{10} \Rightarrow 4\big(\chi(1)+\chi(2)+\chi(5)+\chi(10)\big) \\
\hline
\sqrt{11} \Rightarrow 4\big(\chi(1)+\chi(11)\big) \\
\hline
\sqrt{12} \Rightarrow 4\big(\chi(1)+\chi(2)+\chi(3)+\chi(4)+\chi(6)+\chi(12)\big) \\
\hline
\vdots \\
\hline
\sqrt{R^2}
\end{array}
$$

Figure 8: Illustration of the number of lattice points

So we can say that at the limit

$$\pi R^2 = 4\left(R^2\chi(1) + \frac{R^2}{2}\chi(2) + \frac{R^2}{3}\chi(3) + \frac{R^2}{4}\chi(4) + \dots\right) \tag{3.12}$$

factoring by R^2 and simplifying gives us

$$\pi = 4\left(\chi(1) + \frac{\chi(2)}{2} + \frac{\chi(3)}{3} + \frac{\chi(4)}{4} + \frac{\chi(5)}{5} + \dots\right) \tag{3.13}$$

which is

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots\right) \tag{3.14}$$

which is just the Leibniz series.

The reason I like to do research in mathematics is really because of situations like these. Leibniz demonstrated his series using infinitesimal calculus; continuous mathematics. Yet, this same series can be demonstrated by using discrete mathematics, which Sanderson did. This serves to prove how complementary the mathematical branches are and how the concept of the limit to infinity can also be applied in discrete mathematics.

4 CONCLUSION

To answer the research question "What is the relationship between π, the Leibniz series and the distribution of prime numbers?", we can see the Leibniz series by being the result of the link between π and the distribution of prime numbers; because the way in which prime numbers of the form $4k+1$ can be factorized within the Gaussian integers but those of the form $4k+3$ cannot. Along with the introduction of the Dirichlet character, allow us to achieve this result. In addition, the fact that π is irrational can be considered as a consequence of the configuration of prime numbers in integers. Yet Sanderson's method contains relatively less rigour than the other proofs that exist but it is quite elegant and shows the connection between several domains in mathematics and how tools from several domains can be used to reach a great conclusion in another domain.

However, we can wonder if there are other connections between π and the distribution of prime numbers, and even maybe direct relationships between π and prime numbers. This is a relevant question to ask and indeed, this question is dealt with in analytical number theory and it is certain that several mathematicians have researched this question.

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